On the Neighbourhood Matrices of Circulant Graphs

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There are various matrices that can be defined on graphs such as adjacency matrix, incidence matrix, laplacian matrix, degree matrix etc. These matrices are useful in analysing the structural properties of graphs using algebraic techniques. Recently, a new graph matrix, namely neighbourhood matrix is introduced. What distinguishes neighbourhood matrix from other matrices is that it is actually the product of two graph matrices and it need not be symmetric always.

Neighbourhood matrix introduced in [1] is a square matrix associated with an undirected graph defined using the neighbourhood sets of the vertices. Given a graph G, the neighbourhood matrix, denoted by $\mathcal{NM}(G) = [\eta_{ij}]$ is defined as

$$\eta_{ij} = \begin{cases} -|N_G(i)| & \text{if } i = j \\ |N_G(j) - N_G(i)| & \text{if } (i,j) \in E(G) \\ -|N_G(j) \cap N_G(i)| & \text{if } (i,j) \notin E(G) \end{cases}$$

$\mathcal{NM}(G)$ can also be defined by using the product of adjacency matrix and Laplacian matrix of a graph G.

It is a useful tool for studying networks and for solving network properties quite easily. In [3], the authors defined a sequence of powers of $\mathcal{NM}(G)$ and developed an algorithm to find the shortest path between any pair of nodes in a given graph. Thus, this matrix helps to solve many graph theoretic problems using less time complexity compared to the existing algorithms.

The network $G(n; \pm s_1, \pm s_2, ..., \pm s_k)$ with n nodes and $2k$ links per vertex such that each node i is adjacent to the $2k$ nodes $i \pm s_1, i \pm s_2, ..., i \pm s_k (mod n)$ is called the Circulant graph [2]. Circulant graphs form an important class of graphs that are well studied in literature as good networks due to their high symmetry, regularity and vertex transitivity [4]. Another interesting class of graphs is Cayley graphs, which help in the visualization of a group. Let Γ be a group with generating set S. Then, construct a graph G as follows:

(i) The vertices of G are the elements of Γ

(ii) There is an edge from vertex g to vertex h if $h = gs$ for some $s \epsilon S$. Then the graph G is called the Cayley graph of the group Γ defined by S and is denoted by $\mathrm{Cay}(\Gamma, S)$ [5]. If S is inverse-closed and does not contain the identity, then G is undirected and has no loops. It is interesting to note that all Circulant graphs are $Cay(Z_n, S)$ for some generating set S.

Motivated by these works, in this paper, we have studied the neighbourhood matrices of various Cayley graphs on Z_n by varying the generating set S. Also, a sharp bound on the number of distinct entries in $\mathcal{NM}(G)$, where G is any Circulant graph is also obtained.

References

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